

# Quantum Gowdy model within the new loop quantum cosmology improved dynamics

M Martín-Benito<sup>1,2</sup>, L J Garay<sup>3,2</sup>, G A Mena Marugán<sup>2</sup>

<sup>1</sup>MPI für Gravitationale Physics, Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany.

<sup>2</sup>Instituto de Estructura de la Materia, CSIC, Serrano 121, 28006 Madrid, Spain.

<sup>3</sup>Departamento de Física Teórica II, Universidad Complutense de Madrid, 28040 Madrid, Spain.

E-mail: mercedes@aei.mpg.de, luis.garay@fis.ucm.es, mena@iem.cfmac.csic.es

**Abstract.** The linearly polarized Gowdy  $T^3$  model can be regarded as compact Bianchi I cosmologies with inhomogeneous modes allowed to travel in one direction. We study a hybrid quantization of this model that combines the loop quantization of the Bianchi I background, adopting the improved dynamics scheme put forward by Ashtekar and Wilson-Ewing, with a Fock quantization for the inhomogeneities. The Hamiltonian constraint operator provides a resolution of the cosmological singularity and superselects separable sectors. We analyze the complicated structure of these sectors. In any of them the Hamiltonian constraint provides an evolution equation with respect to the volume of the associated Bianchi I universe, with a well posed initial value problem. This fact allows us to construct the Hilbert space of physical states and to show that we recover the standard quantum field theory for the inhomogeneities.

## 1. Introduction

The Gowdy cosmologies represent a suitable arena to further develop loop quantum cosmology (LQC) [1] and include inhomogeneous settings, since they provide the simplest inhomogeneous cosmological spacetimes [2]. Here, we analyze the particular case of the Gowdy model with three-torus topology and linear polarization. The homogeneous sector of this model (given by the degrees of freedom that parametrize the subset of homogeneous solutions) coincides with the Bianchi I model with three-torus topology. In turn, the inhomogeneities describe a content of linearly polarized gravitational waves traveling in a single direction. In previous studies we carried out the quantization of this system by means of a hybrid procedure [3, 4]. More specifically, we applied a Fock quantization for the inhomogeneous sector, in order to deal with the field complexity, while for the homogeneous sector we employed the quantization of the Bianchi I model performed in LQC [5], with the main aim of achieving a quantum resolution of the cosmological singularity.

The quantization of the Bianchi I model in LQC is subject to several ambiguities. One of them concerns the representation of the curvature tensor of the homogeneous connection in the quantum theory. Different definitions of this object lead to different schemes of quantization. In this respect, Ashtekar and Wilson-Ewing have put forward the quantization of the Bianchi I model within a new *improved dynamics* scheme [6], which seems to be physically more appropriate than the scheme employed in [3, 4, 5]. In this note we summarize the hybrid

quantization of the Gowdy model that results from adopting this new improved dynamics scheme in the representation of the homogeneous sector. We focus on the new features that the use of this scheme introduces, such as the complicated structure of the resulting sectors of superselection (aspect not analyzed in [6]) and the interpretation of the quantum Hamiltonian constraint as an evolution equation with respect to the volume of the associated Bianchi I universe. We will discuss that this interpretation is valid inasmuch as the corresponding initial value problem is well posed: physical solutions are completely determined by a countable set of data given on an initial section of non-vanishing Bianchi I volume. Moreover, this fact allows us to identify the physical Hilbert space with the Hilbert space of initial data. For the details of this work we refer to [7].

## 2. Quantization of the model

The linearly polarized Gowdy  $T^3$  cosmologies are globally hyperbolic spacetimes with three-torus spatial topology and with two axial and hypersurface orthogonal Killing vector fields  $\partial_\sigma$  and  $\partial_\delta$  [2]. We use global coordinates adapted to the symmetries  $\{t, \theta, \sigma, \delta\}$ , with  $\theta, \sigma, \delta \in S^1$ . Then the spatial dependence occurs only in  $\theta$ , and we can expand the field-like variables in Fourier series in that coordinate. After performing a partial gauge fixing [4, 7], the reduced phase space is described by two pairs of canonically conjugate point-particle variables (they do not depend on  $\theta$ ) and by one field  $\xi$ , together with its canonical momentum  $P_\xi$ . Two global constraints remain in the model, the spatial average of the  $\theta$ -momentum constraint (which generates translations in  $S^1$ ) and the spatial average of the densitized Hamiltonian constraint.

We call inhomogeneous sector the set of degrees of freedom given by the nonzero Fourier modes of  $\xi$  and those of  $P_\xi$ . To describe this sector we introduce creation and annihilation-like variables, denoted by  $\{a_m^*, a_m\}$  ( $m \in \mathbb{Z} - \{0\}$ ), defined as if  $\xi$  and  $P_\xi$  were a canonical pair for a free massless scalar field. In the quantum theory we promote them to annihilation and creation operators, with  $[\hat{a}_m, \hat{a}_m^\dagger] = 1$ .  $\mathcal{F}$  will denote the corresponding Fock space. The generator of translations on the circle only involves this inhomogeneous sector. It is not difficult to obtain its quantum counterpart in terms of the basic operators  $\hat{a}_m$  and  $\hat{a}_m^\dagger$ . The states that verify this constraint form a Fock subspace  $\mathcal{F}_p \subset \mathcal{F}$  [4, 7].

In turn, the homogeneous sector is formed by the rest of degrees of freedom: the two point-particle variables and the zero mode of  $\xi$ , together with its momentum. This sector coincides with the phase space of the Bianchi I model. We describe it in the Ashtekar-Barbero formalism. Using a diagonal gauge, the nontrivial components of the densitized triad are  $p_i/4\pi^2$ , with  $i = \theta, \sigma, \delta$ , and  $c_i/2\pi$  are those of the  $su(2)$  connection. They satisfy  $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$ , where  $\gamma$  is the Immirzi parameter and  $G$  is the Newton constant. In order to represent this sector in the quantum theory we choose the kinematical Hilbert space of the Bianchi I model constructed in LQC [5, 6], that we call  $\mathcal{H}_{\text{kin}}^{\text{BI}}$ . We recall that, on  $\mathcal{H}_{\text{kin}}^{\text{BI}}$ , the operators  $\hat{p}_i$  have a discrete spectrum equal to the real line. The corresponding eigenstates,  $|p_\theta, p_\sigma, p_\delta\rangle$ , form an orthonormal basis (in the discrete norm) of  $\mathcal{H}_{\text{kin}}^{\text{BI}}$ . Owing to this discreteness, there is no well defined operator representing the connection, but rather its holonomies. They are computed along straight edges in the fiducial directions. The so-called improved dynamics prescription states that, when writing the curvature tensor in terms of holonomies, we have to evaluate them along edges with a certain minimum dynamical (state dependent) length  $\bar{\mu}_i$ . We use the specific improved dynamics prescription put forward in [6]: all the elementary operators which represent the matrix elements of the holonomies, called  $\hat{\mathcal{N}}_{\bar{\mu}_i}$ , produce a constant shift in the Bianchi I volume of the compact spatial section. In order to simplify the analysis, it is convenient to relabel the basis states in the form  $|v, \lambda_\sigma, \lambda_\delta\rangle$ , where  $v$  is proportional to the volume and such that the operators  $\hat{\mathcal{N}}_{\pm\bar{\mu}_i}$  cause a shift on it equal to  $\pm 1$ , and the parameters  $\lambda_i$  are all equally defined in terms of the corresponding parameters  $p_i$ , and verify that  $v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$ .

The Hamiltonian constraint of the Gowdy model is formed by the Hamiltonian constraint of

the Bianchi I model plus the coupling term which involves both homogeneous and inhomogeneous sectors. Out of the basic operators  $\hat{p}_i$ ,  $\hat{N}_{\vec{\mu}_i}$ ,  $\hat{a}_m$ , and  $\hat{a}_m^\dagger$ , we represent it as an operator  $\hat{\mathcal{C}}$  well defined on a dense domain of the kinematical Hilbert space  $\mathcal{H}_{\text{kin}}^{\text{BI}} \otimes \mathcal{F}$  [7]. We choose a very suitable symmetric factor ordering such that  $\hat{\mathcal{C}}$  decouples the states of  $\mathcal{H}_{\text{kin}}^{\text{BI}} \otimes \mathcal{F}$  with support on  $v = 0$ , namely, the states with vanishing homogeneous volume. Moreover, our operator  $\hat{\mathcal{C}}$  does not relate states with different orientation of any of the eigenvalues of the operators  $\hat{p}_i$ . Owing to this property we can then restrict the homogeneous sector (also in the domain of definition of  $\hat{\mathcal{C}}$ ) to e.g. the space spanned by the states  $|v > 0, \lambda_\sigma > 0, \lambda_\delta > 0\rangle$ . We call the resulting Hilbert space  $\mathcal{H}_{\text{kin}}^{\text{BI},+}$ . Remarkably, for this restriction of the homogeneous sector we do not need to impose any particular boundary condition or appeal to any parity symmetry. Note that both  $\mathcal{H}_{\text{kin}}^{\text{BI}}$  and  $\mathcal{H}_{\text{kin}}^{\text{BI},+}$  are non-separable Hilbert spaces, feature which is not desirable for a physical theory. This problem is overcome by the very action of the Hamiltonian constraint operator. Indeed,  $\hat{\mathcal{C}}$  defined on (a dense domain in)  $\mathcal{H}_{\text{kin}}^{\text{BI},+} \otimes \mathcal{F}$  turns out to leave invariant some separable Hilbert subspaces that provide superselection sectors. Specifically, the homogeneous sector of these superselection sectors is the space spanned by the states  $|v = \varepsilon + 4n, \lambda_\sigma = \lambda_\sigma^* \omega_\varepsilon, \lambda_\delta = \lambda_\delta^* \omega_\varepsilon\rangle$ . Here  $\varepsilon \in (0, 4]$  and  $n \in \mathbb{N}$ ; therefore  $v$  takes support on semilattices of constant step equal to 4 starting in a minimum non-vanishing value  $\varepsilon$ . In addition,  $\lambda_a^*$  ( $a = \sigma, \delta$ ) is some positive real number and  $\omega_\varepsilon$  runs over the subset of  $\mathbb{R}^+$  given by

$$\left\{ \left( \frac{\varepsilon - 2}{\varepsilon} \right)^z \prod_k \left( \frac{\varepsilon + 2m_k}{\varepsilon + 2n_k} \right)^{p_k} \right\},$$

where  $m_k, n_k, p_k \in \mathbb{N}$ , and  $z \in \mathbb{Z}$  when  $\varepsilon > 2$ , while  $z = 0$  otherwise. One can check that indeed this set is dense in  $\mathbb{R}^+$  and countable [7]. Therefore, any of these sectors provide separable Hilbert spaces contained in  $\mathcal{H}_{\text{kin}}^{\text{BI}}$ . We denote them as  $\mathcal{H}_{\varepsilon, \lambda_\sigma^*, \lambda_\delta^*} = \mathcal{H}_\varepsilon \otimes \mathcal{H}_{\lambda_\sigma^*} \otimes \mathcal{H}_{\lambda_\delta^*}$ . Note that the fact that the states with support on  $v = 0$  are removed means that there is no analog in our quantum theory of the cosmological singularities, classically requiring that some  $p_i$  vanish. We thus solve these singularities already at the kinematical level in a very simple way.

We then restrict the study to any of these superselection sectors and define the homogeneous terms of the Hamiltonian constraint operator  $\hat{\mathcal{C}}$  on the span of the states  $|\varepsilon + 4n, \lambda_\sigma^* \omega_\varepsilon, \lambda_\delta^* \omega_\varepsilon\rangle$ . The Hamiltonian constraint turns out to provide a difference equation in the parameter  $v$  [7]. Therefore, if we regard  $v$  as an internal time, the constraint can be interpreted as an evolution equation in it. The condition that this constraint imposes is very complicated and the solutions can be obtained only formally. This is in part owing to the fact that one of the inhomogeneous terms creates in every step of the evolution an infinite number of particles (for a detailed discussion see [3, 4, 7]). Nonetheless, thanks to the precise structure of the superselection sectors for the variables  $\lambda_a$  we can argue that a set of countable data given on the initial section  $v = \varepsilon$  completely determines the physical solutions. This issue is non-trivial. Actually, both the separability of the support of  $\lambda_a$  and the fact that it is dense in  $\mathbb{R}^+$  have been essential to show it (the details can be found in [8]). At the end of the day, the initial value problem is well posed and it is indeed feasible to make the above interpretation of the constraint as an evolution equation in  $v$ .

We can thus identify solutions with initial data. Furthermore, we can construct a(n over) complete set of observables acting on the space of initial data and impose reality conditions on them [7, 8]. This determines a unique inner product that endows the space of initial data with a Hilbert structure. Then, we finally impose the  $S^1$  symmetry and identify the resulting Hilbert space with the physical Hilbert space. The result is that the physical Hilbert space is  $H_{\lambda_\sigma^*} \otimes \mathcal{H}_{\lambda_\delta^*} \otimes \mathcal{F}_p$ , with  $H_{\lambda_\sigma^*} \otimes \mathcal{H}_{\lambda_\delta^*}$  being the physical Hilbert space of the Bianchi I model in LQC [8].

### 3. Conclusions

This work summarizes the quantization in the framework of LQC of the Gowdy  $T^3$  model with linear polarization, which is an inhomogeneous cosmological spacetime with spatial dependence in a single direction  $\theta$ . In order to perform it, first, at the classical level, we fix the gauge partially. The reduced system is subject to two global constraints: the spatial average of the  $\theta$ -momentum constraint and the spatial average of the densitized Hamiltonian constraint. These constraints are imposed à la Dirac in the quantum theory. Moreover, they are represented by means of a hybrid approach, which combines the polymeric techniques of LQC with those of the Fock quantization. The former ones, applied in the representation of the homogeneous sector of the system, allow us to resolve the cosmological singularity in the quantum theory, while the Fock quantization for the inhomogeneities successfully deal with the field complexity.

Employing this hybrid approach we obtain well defined operators for the constraints. For the Hamiltonian constraint operator, we choose a very convenient symmetric factor ordering, such that it allows us to restrict the study of the homogeneous sector to suitable, separable superselection sectors. In any of them, the Hamiltonian constraint provides a difference equation in the volume of the homogeneous sector of the Gowdy model, which coincides with the Bianchi I model. We have interpreted this volume as an internal time and the Hamiltonian constraint as an evolution equation in that time. We have argued that this interpretation is valid, inasmuch as the corresponding initial value problem is well posed. In fact, a countable dense set of initial data completely specifies the physical solutions, and then we can characterize the physical Hilbert space as a Hilbert space of initial data. The physical inner product is fixed by demanding reality conditions on a complete set of observables and imposing the  $S^1$  symmetry. We conclude that the physical Hilbert space is the physical Hilbert space of the Bianchi I model in LQC tensor product a Fock space  $\mathcal{F}_p$ .

It is important to note that, if the model is totally deparametrized and quantized as in standard quantum field theory, it admits a unique Fock quantization with unitary dynamics and such that the vacuum is invariant under  $S^1$  translations [9, 10, 11]. The physical Hilbert space for the inhomogeneities, namely  $\mathcal{F}_p$ , is actually equivalent to that obtained for them in the standard quantization of the deparametrized model. Therefore we recover the standard description of the inhomogeneities, which can be seen as degrees of freedom propagating over the polymerically quantized Bianchi I background. This result, although expected, is not trivial since homogeneous and inhomogeneous sectors have been quantized with very different methods and they are coupled in a very involved way by the Hamiltonian constraint.

### Acknowledgments

This work was in part supported by the Spanish MICINN Project No. FIS2008-06078-C03-03 and the Consolider-Ingenio Program CPAN No. CSD2007-00042. M. M-B has been supported by CSIC and the European Social Fund under the grant I3P-BPD2006.

### References

- [1] Bojowald M 2008 *Living Rev. Rel.* **11** 4
- [2] Gowdy R H 1974 *Ann. Phys.* **83** 203
- [3] Martín-Benito M, Garay L J and Mena Marugán G A 2008 *Phys. Rev. D* **78** 083516
- [4] Mena Marugán G A and Martín-Benito M 2009 *Int. J. Mod. Phys. A* **24** 2820
- [5] Martín-Benito M, Mena Marugán G A and Pawłowski T 2008 *Phys. Rev. D* **78** 064008
- [6] Ashtekar A and Wilson-Ewing E 2009 *Phys. Rev. D* **79** 083535
- [7] Garay L J, Martín-Benito M and Mena Marugán G A 2010 *Phys. Rev. D* **82** 044048
- [8] Martín-Benito M, Mena Marugán G A and Wilson-Ewing E 2010 *Phys. Rev. D* **82** 084012
- [9] Corichi A, Cortez J and Mena Marugán G A 2006 *Phys. Rev. D* **73** 084020
- [10] Corichi A, Cortez J, Mena Marugán G A and Velhinho J M 2006 *Classical Quantum Gravity* **23** 6301
- [11] Cortez J, Mena Marugán G A and Velhinho J M 2007 *Phys. Rev. D* **75** 084027